3 分数阶方程的 block-by-block 算法的最优阶收敛性分析

p-block-by-block 方法是一个对积分方程的线性多步法[34]. 此方法导出了一个在 $m+1$ 个步长块上有 p 个未知量 $u^{pm+1}, u^{pm+2}, \cdots,$ 和 u^{pm+p} ,解这 p 个未知量是非常困难的，尤其当 p 个未知量是空间变量 x 的函数时. 1985 年，Linz[35] 对非线性 Volterra 积分方程首先提出 block-by-block 方法；2006 年，Kumar 和 Agrawal[10]把 block-by-block 方法应用到求解 FODEs 的一组初值问题，数值例子显示了此算法的稳定性，但是遗憾的是没有给出该算法的收敛性分析. 2012 年，Huang 等人[11]证明了此方法的收敛阶至少是 3 阶，但是该文的数值例子显示了该算法的最优收敛阶：当 $0<\alpha\leqslant 1$ 时，收敛阶为 $3+a$ 阶；当 $\alpha>1$ 时，收敛阶为 4 阶. 因此，本章利用文献[36]的技巧通过对误差估计进行精细估计，对该算法的最优收敛阶给出严格证明.

3.1 block-by-block 算法的构造

本章中，我们仍然考虑初值问题（2.1）-（2.2）. 由 Kumar 和 Agrawal[10] 构造的 block-by-block 方法描述如下：

结合（2.7），（2.11），我们有如下的式子：

$$\begin{cases} u_{2m+1} = g_{2m+1} + \sum_{k=0}^{m-1}[A_{2m+1}^{0,k} f_{2k} + A_{2m+1}^{1,k} f_{2k+1} + A_{2m+1}^{2,k} f_{2k+2}] \\ \qquad + A_{2m+1}^{0,m} f_{2m} + A_{2m+1}^{1,m} f_{2m+1} + A_{2m+1}^{2,m} f_{2m+2}, \\ u_{2m+2} = g_{2m+2} + \sum_{k=0}^{m}[A_{2m+2}^{0,k} f_{2k} + A_{2m+2}^{1,k} f_{2k+1} + A_{2m+2}^{2,k} f_{2k+2}], \\ m = 1, \cdots, N-1. \end{cases} \tag{3.1}$$

3.2 辅助结果

引理 3.1 设：

$$\beta_k=(k+1)^{\alpha+1}-k^{\alpha+1}-\frac{6}{\alpha+2}[(k+1)^{\alpha+2}+k^{\alpha+2}]+\frac{12}{(\alpha+2)(\alpha+3)}[(k+1)^{\alpha+3}-k^{\alpha+3}],$$

则对所有的正整数 m，存在一个依赖于 α 的常数 C，当 $0<\alpha\leqslant 1$ 时，满足：

$$\sum_{k=0}^{m}|\beta_k|\leqslant C, \qquad (3.2)$$

当 $\alpha>1$ 时，有：

$$\sum_{k=0}^{m}|\beta_k|\leqslant Cm^{\alpha-1}. \qquad (3.3)$$

证明 首先，我们有：

$$\beta_0=1-\frac{6}{\alpha+2}+\frac{12}{(\alpha+2)(\alpha+3)}=\frac{\alpha(\alpha-1)}{(\alpha+2)(\alpha+3)},$$

$$\begin{aligned}\beta_1&=2^{\alpha+1}-1-\frac{6}{\alpha+2}(2^{\alpha+1}+1)+\frac{12}{(\alpha+2)(\alpha+3)}(2^{\alpha+3}-1)\\&=\frac{2^{\alpha+1}(\alpha^2-7\alpha+18)-(\alpha^2+11\alpha+36)}{(\alpha+2)(\alpha+3)}.\end{aligned}$$

因此，对于所有的 $\alpha>0$，有下式成立：

$$|\beta_0|+|\beta_1|\leqslant C<\infty,$$

这里 C 是仅依赖于 $2^{\alpha-1}$ 的常数.

当 $k\geqslant 2$ 时，可得：

$$|\beta_k|=k^{\alpha+1}\left|\left(1+\frac{1}{k}\right)^{\alpha+1}-1-\frac{6k}{\alpha+2}\left[\left(1+\frac{1}{k}\right)^{\alpha+2}+1\right]\right.$$

$$+\frac{12k^2}{(\alpha+2)(\alpha+3)}\left[\left(1+\frac{1}{k}\right)^{\alpha+3}-1\right]\Bigg|$$

$$=k^{\alpha+1}\left|\sum_{i=1}^{\infty}\frac{\prod_{j=1}^{i}(\alpha+2-j)}{i!k^i}-\frac{6k}{\alpha+2}\left[2+\sum_{i=1}^{\infty}\frac{\prod_{j=1}^{i}(\alpha+32-j)}{i!k^i}\right]\right.$$

$$\left.+\frac{12k^2}{(\alpha+2)(\alpha+3)}\sum_{i=1}^{\infty}\frac{\prod_{j=1}^{i}(\alpha+4-j)}{i!k^i}\right|$$

$$=k^{\alpha+1}\left|\sum_{i=1}^{\infty}\frac{\prod_{j=1}^{i}(\alpha+2-j)}{i!k^i}-6\sum_{i=1}^{\infty}\frac{\prod_{j=1}^{i}(\alpha+2-j)}{(i+1)!k^i}+12\sum_{i=1}^{\infty}\frac{\prod_{j=1}^{i}(\alpha+2-j)}{(i+2)!k^i}\right|$$

$$=k^{\alpha+1}\left|(1+\alpha)\alpha\sum_{i=1}^{\infty}\frac{\prod_{j=1}^{i}(\alpha-1-j)}{(i+5)!k^{i+3}}(i+2)(i+1)\right|.$$

我们先证明（3.2）. 当 $\alpha=0$ 时，对所有的 $k\geqslant 2$，有：

$$\beta_k=(k+1)^{0+1}-k^{0+1}-\frac{6}{0+2}[(k+1)^{0+2}+k^{0+2}]+\frac{12}{(0+2)(0+3)}[(k+1)^{0+3}-k^{0+3}]$$

$$=(k+1)-k-3[(k+1)^2+k^2]+2[(k+1)^3-k^3]$$

$$=1-3(2k^2+2k+1)+2(3k^2+3k+1)$$

$$=0.$$

当 $\alpha=1$ 时，对所有的 $k\geqslant 2$，有：

$$\beta_k = (k+1)^{1+1} - k^{1+1} - \frac{6}{1+2}[(k+1)^{1+2} + k^{1+2}] + \frac{12}{(1+2)(1+3)}[(k+1)^{1+3} - k^{1+3}]$$
$$= (k+1)^2 - k^2 - 2[(k+1)^3 + k^3] + [(k+1)^4 - k^4]$$
$$= 2k+1 - 2(2k^3 + 3k^2 + 3k + 1) + (4k^3 + 6k^2 + 4k + 1)$$
$$= 0.$$

因此，当 $\alpha = 0$ 或者 1 时，对所有的 $k \geqslant 2$，都有 $\beta_k = 0$，因而（3.2）显然成立；

当 $0 < \alpha < 1$ 时，我们将证明级数 $\sum_{k=2}^{\infty} |\beta_k|$ 收敛. 我们知道，当 $s > 1$ 时，级数 $\sum_{k=2}^{\infty} \frac{1}{k^s}$ 是收敛的，所以只需证明足够大的 k，$|\beta_k| \leqslant C\frac{1}{k^s}$ 即可. 事实上，对 $k \geqslant 2$，

$$|\beta_k| = \left|\frac{(1+\alpha)\alpha(\alpha-1)}{60}\right| \frac{1}{k^{2-\alpha}} \left|1 + \frac{\alpha-2}{2}\frac{1}{k} + \frac{(\alpha-2)(\alpha-3)}{7}\frac{1}{k^2} + \cdots\right|$$
$$\leqslant \left|\frac{(1+\alpha)\alpha(\alpha-1)}{60}\right| \frac{1}{k^{2-\alpha}} \left(1 + \frac{1}{k} + \frac{1}{k^2} + \cdots\right)$$
$$\leqslant \frac{2\left(1+\frac{\sqrt{3}}{3}\right)\frac{\sqrt{3}}{3}\left(1-\frac{\sqrt{3}}{3}\right)}{60} \frac{1}{k^{2-\alpha}}$$
$$= \frac{\sqrt{3}}{135} \frac{1}{k^{1+(1-\alpha)}}.$$

因此，级数 $\sum_{k=1}^{\infty} |\beta_k|$ 是收敛的，也即对于所有的 $m > 0$，（3.2）都是成立的.

接下来，我们将证明（3.3）. 当 $\alpha > 1$ 时，记

$$q_i := \frac{\prod_{j=0}^{i}(\alpha - 1 - j)}{(i+5)!}(i+2)(i+1).$$

因此对 $k \geqslant 2$，有：

$$\left|\beta_k\right| = \frac{1}{k^{2-\alpha}}\left|(1+\alpha)\alpha\sum_{i=0}^{\infty}\frac{\prod_{j=0}^{i}(\alpha-1-j)}{(i+5)!k^i}(i+2)(i+1)\right|$$

$$= \frac{(1+\alpha)\alpha}{k^{2-\alpha}}\left|\sum_{i=0}^{\infty}\frac{q_i}{k^i}\right|.$$

当 $k \geqslant 2$ 时，我们将证明级数 $\sum_{i=0}^{\infty}\frac{q_i}{k^i}$ 是收敛的. 注意到，对于任意的 i ，$\left|q_i\right|$ 是一致有界的，尽管此界可能依赖于 α. 因此，当 $k \geqslant 2$ 时，级数 $\sum_{i=0}^{\infty}\frac{q_i}{k^i}$ 是一致收敛的，即存在一个与 k 无关的常数 C_1 ，使得 $\left|\sum_{i=0}^{\infty}\frac{q_i}{k^i}\right|$ 成立. 我们有：

$$\sum_{k=2}^{m}\left|\beta_k\right| \leqslant C_1\sum_{k=2}^{m}\frac{(1+\alpha)\alpha}{k^{2-\alpha}} \leqslant C_1(1+\alpha)\alpha\int_1^m k^{\alpha-2}\mathrm{d}k$$

$$= \frac{C_1(1+\alpha)\alpha}{\alpha-1}(m^{\alpha-1}-1) \leqslant Cm^{\alpha-1}.$$

因此，我们得到（3.3）. 引理 3.1 证明完毕.

3.3 截断误差的估计

现在，我们将对格式（3.1）的截断误差作一个估计. 首先，我们分析奇数层的局部截断误差. 我们定义 $2m+1$ 层的截断误差为：

$$r_{2m+1}(\Delta t) := u(t_{2m+1}) - \bar{u}_{2m+1}, \tag{3.4}$$

这里 $\bar{u}_{2m+1}$ 是 $u(t_{2m+1})$ 的一个逼近. 将精确解代入（2.7），得

$$\begin{aligned}\bar{u}_{2m+1} &= g_{2m+1} \\ &+ \sum_{k=0}^{m-1}[A_{2m+1}^{0,k}f(t_{2k},u(t_{2k})) + A_{2m+1}^{1,k}f(t_{2k+1},u(t_{2k+1})) + A_{2m+1}^{2,k}f(t_{2k+2},u(t_{2k+2}))] \\ &+ A_{2m+1}^{0,m}f(t_{2m},u(t_{2m})) + A_{2m+1}^{1,m}f(t_{2m+1},u(t_{2m+1})) + A_{2m+1}^{2,m}f(t_{2m+2},u(t_{2m+2})).\end{aligned} \tag{3.5}$$

因此，对于 $r_{2m+1}(\Delta t)$, 有如下的估计.

引理 3.2　设 $r_{2m+1}(\Delta t)$ 是（3.4）中定义的截断误差. 设 $f(\cdot,u(\cdot))\in C^4[0,T]$. 因此，当 $0<\alpha\leqslant 1$ 时，它满足：

$$\left|r_{2m+1}(\Delta t)\right|\leqslant C\Delta t^{3+\alpha},$$

当 $\alpha>1$ 时，有：

$$\left|r_{2m+1}(\Delta t)\right|\leqslant C\Delta t^{4}.$$

证明　结合（2.5），（2.7）和（3.5），我们有：

$$\begin{aligned}
r_{2m+1}(\Delta t)=u(t_{2m+1})-\Bigg\{&g_{2m+1}\\
&+\sum_{k=0}^{m-1}[A_{2m+1}^{0,k}f(t_{2k},u(t_{2k}))+A_{2m+1}^{1,k}f(t_{2k+1},u(t_{2k+1}))+A_{2m+1}^{2,k}f(t_{2k+2},u(t_{2k+2}))]\\
&+f(t_{2m},u(t_{2m}))\omega_{2m+1}^{0,m}+\Bigg[\frac{3}{8}f(t_{2m},u(t_{2m}))+\frac{3}{4}f(t_{2m+1},u(t_{2m+1}))\\
-\frac{1}{8}f(t_{2m+2},u(t_{2m+2}))\Bigg]&\omega_{2m+1}^{1,m}+f(t_{2m+1},u(t_{2m+1}))\omega_{2m+1}^{2,m}\Bigg\}\\
=g(t_{2m+1})&+\frac{1}{\Gamma(\alpha)}\Bigg[\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,u(\tau))\mathrm{d}\tau\\
&+\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}f(\tau,u(\tau))\mathrm{d}\tau\Bigg]\\
&-\Bigg\{g_{2m+1}+\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\Bigg[f(t_{2k},u(t_{2k}))\int_{2k}^{2k+2}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{0,k}\mathrm{d}\tau\\
&+f(t_{2k+1},u(t_{2k+1}))\int_{2k}^{2k+2}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{1,k}\mathrm{d}\tau\\
&+f(t_{2k+2},u(t_{2k+2}))\int_{2k}^{2k+2}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{2,k}\mathrm{d}\tau\Bigg]\\
&+\frac{1}{\Gamma(\alpha)}f(t_{2m},u(t_{2m}))\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{0,m}\mathrm{d}\tau\\
&+\Bigg[\frac{3}{8}f(t_{2m},u(t_{2m}))+\frac{3}{4}f(t_{2m+1},u(t_{2m+1}))-\frac{1}{8}f(t_{2m+2},u(t_{2m+2}))\Bigg]\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{1,m}\mathrm{d}\tau\\
&+f(t_{2m+1},u(t_{2m+1}))\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\varphi_{2,m}\mathrm{d}\tau\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\{f(\tau,u(\tau))-[f(t_{2k},u(t_{2k}))\varphi_{0,k}(\tau)\\
&\quad+f(t_{2k+1},u(t_{2k+1}))\varphi_{1,k}(\tau)+f(t_{2k+2},u(t_{2k+2}))\varphi_{2,k}(\tau)]\}\mathrm{d}\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\Big\{f(\tau,u(\tau))-[f(t_{2m},u(t_{2m}))\varphi_{0,m}(\tau)+f(t_{2m+\frac{1}{2}},u(t_{2m+\frac{1}{2}}))\varphi_{1,m}(\tau)\\
&\quad+f(t_{2m+1},u(t_{2m+1}))\varphi_{2,m}(\tau)]+\Big[f(t_{2m+\frac{1}{2}},u(t_{2m+\frac{1}{2}}))\\
&\quad-\left(\frac{3}{8}f(t_{2m},u(t_{2m}))+\frac{3}{4}f(t_{2m+1},u(t_{2m+1}))-\frac{1}{8}f(t_{2m+2},u(t_{22m+2}))\right)\Big]\varphi_{1,m}(\tau)\Big\}\mathrm{d}\tau\\
&=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}R_{2k}(\tau)\mathrm{d}\tau\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}(R_{2m-1}(\tau)+R_{2m}(\tau)\varphi_{1,m}(\tau))\mathrm{d}\tau,
\end{aligned}
$$

这里：

$$
\begin{aligned}
R_{2k}(\tau)&=f(\tau,u(\tau))-f(t_{2k},u(t_{2k}))\varphi_{0,k}(\tau)\\
&\qquad-f(t_{2k+1},u(t_{2k+1}))\varphi_{1,k}(\tau)-f(t_{2k+2},u(t_{2k+2}))\varphi_{2,k}(\tau),\\
R_{2m-1}(\tau)&=f(\tau,u(\tau))-f(t_{2m},u(t_{2m}))\varphi_{0,m}(\tau)-f(t_{2m+\frac{1}{2}},u(t_{2m+\frac{1}{2}}))\varphi_{1,m}(\tau)\\
&\qquad-f(t_{2m+1},u(t_{2m+1}))\varphi_{2,m}(\tau),\\
R_{2m}(\tau)&=f(t_{2m+\frac{1}{2}},u(t_{2m+\frac{1}{2}}))-\left(\frac{3}{8}f(t_{2m},u(t_{2m})\right.\\
&\qquad\left.+\frac{3}{4}f(t_{2m+1},u(t_{2m+1}))-\frac{1}{8}f(t_{2m+2},u(t_{22m+2}))\right).
\end{aligned}
$$

利用 Taylor 定理，对于所有的 $\tau\in[t_{2k},t_{2k+2}]$，存在 $\xi_k(\tau)\in[t_{2k},t_{2k+2}]$，使得：

$$
R_{2k}(\tau)=\frac{f^{(3)}(\xi_k(\tau),u(\xi_k(\tau)))}{3!}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2}),\forall\tau\in[t_{2k},t_{2k+2}].
$$

和对于所有的 $\tau\in[t_{2m},t_{2m+1}]$，存在 $\xi_1(\tau),\xi(\tau)\in[t_{2m},t_{2m+1}]$，使得：

$$R_{2m-1}(\tau)=\frac{f^{(3)}(\xi_1(\tau),u(\xi_1(\tau)))}{3!}(\tau-t_{2m})(\tau-t_{2m+\frac{1}{2}})(\tau-t_{2m+1}),$$

$$R_{2m}(\tau)=\frac{1}{16}\Delta t^3 f^{(3)}(\xi(\tau),u(\xi(\tau))).$$

因此，我们有：

$$\begin{aligned}
r_{2m+1}(\Delta t)=&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\xi_k(\tau),u(\xi_k(\tau)))}{3!}\\
&\cdot(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\xi_1(\tau),u(\xi_1(\tau)))}{3!}\\
&\cdot(\tau-t_{2m})(\tau-t_{2m+\frac{1}{2}})(\tau-t_{2m+1})\mathrm{d}\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\frac{1}{16}\Delta t^3 f^{(3)}(\xi(\tau),u(\xi(\tau)))\varphi_{1,m}(\tau)\mathrm{d}\tau.
\end{aligned} \tag{3.6}$$

下面，我们开始逐项估计（3.6）的右端项. 我们将第一项记为 $R1$，有

$$\begin{aligned}
|R1|=&\left|\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\xi_k(\tau),u(\xi_k(\tau)))}{3!}\right.\\
&\left.\cdot(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|\\
\leqslant&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left\{\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\tilde{\xi}_k,u(\tilde{\xi}_k))}{3!}\right.\right.\\
&\left.\cdot(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|\\
&+\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\xi_k(\tau),u(\xi_k(\tau)))-f^{(3)}(\tilde{\xi}_k,u(\tilde{\xi}_k))}{3!}\right.\\
&\left.\left.\cdot(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|\right\},
\end{aligned} \tag{3.7}$$

其中 $\tilde{\xi}_k=t_{2k+1}$. 由（3.7）右端的第一项，我们得：

$$\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\frac{f^{(3)}(\tilde{\xi}_k,u(\tilde{\xi}_k))}{3!}\cdot(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|$$

$$\leqslant\frac{M_1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|$$

$$\leqslant\frac{M_1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+1}}(t_{2m+1}-\tau)^{\alpha-1}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right.$$

$$\left.+\int_{t_{2k+1}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|$$

$$=\frac{M_1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|(t_{2m+1}-\tilde{\tau}_k)^{\alpha-1}\int_{t_{2k}}^{t_{2k+1}}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right.$$

$$\left.+(t_{2m+1}-\bar{\tau}_k)^{\alpha-1}\int_{t_{2k+1}}^{t_{2k+2}}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|$$

$$=\frac{M_1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|(t_{2m+1}-\tilde{\tau}_k)^{\alpha-1}\cdot\frac{1}{4}\Delta t^4+(t_{2m+1}-\bar{\tau}_k)^{\alpha-1}\cdot\frac{1}{4}\Delta t^4\right|$$

$$=\frac{M_1\Delta t^4}{4\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|(t_{2m+1}-\tilde{\tau}_k)^{\alpha-1}+(t_{2m+1}-\bar{\tau}_k)^{\alpha-1}\right|$$

$$=\frac{M_1\Delta t^4}{4\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|(\alpha-1)(t_{2m+1}-\hat{\tau}_k)^{\alpha-2}(\bar{\tau}_k-\tilde{\tau}_k)\right|$$

$$\leqslant\frac{M_1\Delta t^4}{4\Gamma(\alpha)}|\alpha-1|\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-2}\mathrm{d}\tau\right|$$

$$\leqslant\frac{M_1\Delta t^4}{4\Gamma(\alpha)}|\alpha-1|\left|\int_{t_0}^{t_{2m}}(t_{2m+1}-\tau)^{\alpha-2}\mathrm{d}\tau\right|$$

$$=\frac{M_1\Delta t^4}{4\Gamma(\alpha)}|\alpha-1|\left|-\frac{1}{\alpha-1}(t_{2m+1}-\tau)^{\alpha-1}\Big|_{t_0}^{t_{2m}}\right|$$

$$\leqslant\frac{M_1\Delta t^4}{4\Gamma(\alpha)}\left(\left|(t_{2m+1}-t_{2m})^{\alpha-1}\right|+\left|(t_{2m+1}-t_0)^{\alpha-1}\right|\right)$$

$$=\frac{M_1\Delta t^4}{4\Gamma(\alpha)}\left(\Delta t^{\alpha-1}+t_{2m+1}^{\alpha-1}\right)$$

$$=\frac{M_1\Delta t^{\alpha+3}}{4\Gamma(\alpha)}+\frac{M_1\Delta t^4}{4\Gamma(\alpha)}t_{2m+1}^{\alpha-1}, \tag{3.8}$$

其中：

$$\tilde{\tau}_k \leqslant \hat{\tau}_k \leqslant \overline{\tau}_k,$$
$$t_{2k} \leqslant \tilde{\tau}_k \leqslant t_{2k+1},$$
$$t_{2k+1} \leqslant \overline{\tau}_k \leqslant t_{2k+2},$$
$$M_1 = \sup_{t\in[0,T]}\left|f^{(3)}(t,u(t))\right|.$$

由式（3.7）右端的第二项，得：

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\cdot\frac{f^{(3)}(\xi_k(\tau),u(\xi_k(\tau)))-f^{(3)}(\tilde{\xi}_k,u(\tilde{\xi}_k))}{3!}\right.\\
&\left.\cdot(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|\\
&\leqslant\frac{M_2\Delta t}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\left|(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|\\
&\leqslant\frac{M_2\Delta t^4}{\Gamma(\alpha)}\sum_{k=0}^{m-1}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau\\
&\leqslant\frac{M_2\Delta t^4}{\Gamma(\alpha)}\int_{t_0}^{t_{2m}}(t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau\\
&\leqslant\frac{M_2\Delta t^4}{\Gamma(\alpha)}\left|-\frac{1}{\alpha}(t_{2m+1}-\tau)^{\alpha}\Big|_{t_0}^{t_{2m}}\right|\\
&\leqslant\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}\left(\left|(t_{2m+1}-t_{2m})^{\alpha}\right|+\left|(t_{2m+1}-t_0)^{\alpha}\right|\right)\\
&=\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}\left(\Delta t^{\alpha}+t_{2m+1}^{\alpha}\right)\\
&\leqslant\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}\left(\Delta t^{\alpha}+T^{\alpha}\right)\\
&=\frac{M_2\Delta t^{\alpha+4}}{\alpha\Gamma(\alpha)}+\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}T^{\alpha},
\end{aligned}\qquad (3.9)$$

其中：

$$M_2 = \sup_{t\in[0,T]}\left|f^{(4)}(t,u(t))\right|$$

在上式的估计中，我们用到了以下不等式：

$$\left|\frac{f^{(3)}(\xi_k(\tau),u(\xi_k(\tau)))-f^{(3)}(\tilde{\xi}_k,u(\tilde{\xi}_k))}{3!}\right|\leqslant M_2\Delta t,\text{当}\tilde{\xi}_k=t_{2k+1},\forall\tau\in[t_{2k},t_{2k+2}].$$

将（3.8）-（3.9）代入到（3.7），得：

$$|R1|\leqslant\frac{M_1\Delta t^{\alpha+3}}{4\Gamma(\alpha)}+\frac{M_1\Delta t^4}{4\Gamma(\alpha)}t_{2m+1}^{\alpha-1}+\frac{M_2\Delta t^{\alpha+4}}{\alpha\Gamma(\alpha)}+\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}T^{\alpha}. \tag{3.10}$$

我们记第二项为 $R2$，有如下估计：

$$\begin{aligned}|R2|&\leqslant\frac{1}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\left|\frac{f^{(3)}(\xi_1(\tau),u(\xi_1(\tau)))}{3!}\right.\\&\qquad\left.\cdot(\tau-t_{2m})(\tau-t_{2m+\frac{1}{2}})(\tau-t_{2m+1})\right|\mathrm{d}\tau\\&\leqslant\frac{M_1\Delta t^3}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau\\&=\frac{M_1\Delta t^{\alpha+3}}{\alpha\Gamma(\alpha)}.\end{aligned} \tag{3.11}$$

我们记第三项为 $R3$，有如下估计：

$$\begin{aligned}|R3|&\leqslant\frac{M_1\Delta t^3}{16\Gamma(\alpha)}\left|\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\frac{(\tau-t_{2m})(\tau-t_{2m+1})}{-\frac{1}{4}\Delta t^2}\mathrm{d}\tau\right|\\&\leqslant\frac{M_1\Delta t^3}{\Gamma(\alpha)}\int_{t_{2m}}^{t_{2m+1}}(t_{2m+1}-\tau)^{\alpha-1}\mathrm{d}\tau\\&\leqslant\frac{M_1\Delta t^{\alpha+3}}{\alpha\Gamma(\alpha)}.\end{aligned} \tag{3.12}$$

联立（3.6），（3.10）~（3.12），可得：

$$|r_{2m+1}(\Delta t)|\leqslant\frac{M_1\Delta t^{\alpha+3}}{4\Gamma(\alpha)}+\frac{M_1\Delta t^4}{4\Gamma(\alpha)}t_{2m+1}^{\alpha-1}+\frac{M_2\Delta t^{\alpha+4}}{\alpha\Gamma(\alpha)}+\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}T^{\alpha}+\frac{2M_1\Delta t^{\alpha+3}}{\alpha\Gamma(\alpha)}. \tag{3.13}$$

当 $0<\alpha\leqslant1$ 时，由（3.13），我们有

$$\begin{aligned}
\left|r_{2m+1}(\Delta t)\right| &\leqslant \frac{M_1\Delta t^{\alpha+3}}{4\Gamma(\alpha)}+\frac{M_1\Delta t^4}{4\Gamma(\alpha)}t_{2m+1}^{\alpha-1}+\frac{M_2\Delta t^{\alpha+4}}{\alpha\Gamma(\alpha)}+\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}T^\alpha+\frac{2M_1\Delta t^{\alpha+3}}{\alpha\Gamma(\alpha)} \\
&= \frac{M_1\Delta t^{\alpha+3}}{4\Gamma(\alpha)}+\frac{M_1\Delta t^4}{4\Gamma(\alpha)}[(2m+1)\Delta t]^{\alpha-1}+\frac{M_2\Delta t^{\alpha+4}}{\alpha\Gamma(\alpha)}+\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}T^\alpha+\frac{2M_1\Delta t^{\alpha+3}}{\alpha\Gamma(\alpha)} \\
&= \frac{M_1\Delta t^{\alpha+3}}{4\Gamma(\alpha)}+\frac{M_1(2m+1)^{\alpha-1}}{4\Gamma(\alpha)}\Delta t^{\alpha+3}+\frac{M_2\Delta t^{\alpha+4}}{\alpha\Gamma(\alpha)}+\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}T^\alpha+\frac{2M_1\Delta t^{\alpha+3}}{\alpha\Gamma(\alpha)} \\
&\leqslant C\Delta t^{\alpha+3}.
\end{aligned}$$

这里 C 仅依赖于 M_1, M_2, α 和 T.

当 $\alpha > 1$ 时，我们有：

$$\begin{aligned}
\left|r_{2m+1}(\Delta t)\right| &\leqslant \frac{M_1\Delta t^{\alpha+3}}{4\Gamma(\alpha)}+\frac{M_1\Delta t^4}{4\Gamma(\alpha)}T^{\alpha-1}+\frac{M_2\Delta t^{\alpha+4}}{\alpha\Gamma(\alpha)}+\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}T^\alpha+\frac{2M_1\Delta t^{\alpha+3}}{\alpha\Gamma(\alpha)} \\
&\leqslant C\Delta t^4.
\end{aligned}$$

引理 3.2 证明完毕.

类似于奇数层的截断误差，我们定义偶数层的截断误差：

$$r_{2m+2}(\Delta t) := u(t_{2m+2}) - \bar{u}_{2m+2}, \tag{3.14}$$

这里 $\bar{u}_{2m+2}$ 是 $u(t_{2m+2})$ 的一个逼近. 将精确解代入（2.11）即可得到，下面的引理给出了 $r_{2m+2}(\Delta t)$ 的估计.

引理 3.3　设 $r_{2m+2}(\Delta t)$ 是（3.14）中定义的截断误差. 设 $f(\cdot, u(\cdot)) \in C^4[0,T]$. 因此，当 $0 < \alpha \leqslant 1$ 时，它满足：

$$\left|r_{2m+2}(\Delta t)\right| \leqslant C\Delta t^{3+\alpha},$$

当 $\alpha > 1$ 时，有：

$$\left|r_{2m+2}(\Delta t)\right| \leqslant C\Delta t^4.$$

证明　类似于引理 3.2 的证明，我们有：

$$
\begin{aligned}
r_{2m+2}(\Delta t) &= u(t_{2m+2}) - \overline{u}_{2m+2} \\
&= g(t_{2m+2}) + \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}f(\tau,u(\tau))\mathrm{d}\tau \\
&\quad - g_{2m+2} - \sum_{k=0}^{m}[A_{2m+2}^{0,k}f(t_{2k},u(t_{2k})) \\
&\quad + A_{2m+2}^{1,k}f(t_{2k+1},u(t_{2k+1})) + A_{2m+2}^{2,k}f(t_{2k+2},u(t_{2k+2}))] \\
&= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\big[f(\tau,u(\tau)) - \psi_{0,k}(\tau)f(t_{2k},u(t_{2k})) \\
&\quad - \psi_{1,k}(\tau)f(t_{2k+1},u(t_{2k+1})) - \psi_{2,k}(\tau)f(t_{2k+2},u(t_{2k+2}))\big]\mathrm{d}\tau \\
&= \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\frac{f^{(3)}(\theta_k(\tau),u(\theta_k(\tau)))}{3!} \\
&\quad \cdot(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau,
\end{aligned}
$$

这里 $\theta_k(\tau)\in[t_{2k},t_{2k+2}]$.

使用（3.7）的估计技巧，我们有：

$$
\begin{aligned}
|r_{2m+2}(\Delta t)| \leqslant \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\Bigg\{&\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\frac{f^{(3)}(\hat{\theta}_k,u(\hat{\theta}_k))}{3!}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right| \\
&+\Bigg|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\frac{f^{(3)}(\theta_k(\tau),u(\theta_k(\tau)))-f^{(3)}(\hat{\theta}_k,u(\hat{\theta}_k))}{3!} \\
&\cdot(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\Bigg|\Bigg\},
\end{aligned}
\tag{3.15}
$$

其中 $\hat{\theta}_k=t_{2k+1}$. 由（3.15）右端的第一项，我们得到：

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\frac{f^{(3)}(\hat{\theta}_k,u(\hat{\theta}_k))}{3!}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right| \\
&=\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\left|\frac{f^{(3)}(\hat{\theta}_k,u(\hat{\theta}_k))}{3!}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right| \\
&\leqslant\frac{M_1}{\Gamma(\alpha)}\sum_{k=0}^{m}\left|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2^{\alpha+2}\Delta t^{\alpha+3}M_1}{\Gamma(\alpha+2)}\sum_{k=0}^{m}\Big|(m-k+1)^{\alpha+1}-(m-k)^{\alpha+1} \\
&\quad -\frac{6}{\alpha+2}[(m-k+1)^{\alpha+2}+(m-k)^{\alpha+2}] \\
&\quad +\frac{12}{(\alpha+2)(\alpha+3)}[(m-k+1)^{\alpha+3}-(m-k)^{\alpha+3}]\Big| \\
&= \frac{2^{\alpha+2}\Delta t^{\alpha+3}M_1}{\Gamma(\alpha+2)}\sum_{k=0}^{m}\Big|(k+1)^{\alpha+1}-k^{\alpha+1}-\frac{6}{\alpha+2}[(k+1)^{\alpha+2}+k^{\alpha+2}] \\
&\quad +\frac{12}{(\alpha+2)(\alpha+3)}[(k+1)^{\alpha+3}-k^{\alpha+3}]\Big| \\
&= \frac{2^{\alpha+2}\Delta t^{\alpha+3}M_1}{\Gamma(\alpha+2)}\sum_{k=0}^{m}|\beta_k|, \qquad (3.16)
\end{aligned}
$$

其中 $M_1=\sup_{t\in[0,T]}\left|f^{(3)}(t,u(t))\right|$, β_k 如引理 3.1 所定义.

根据（3.15）右端的第二项，得：

$$
\begin{aligned}
&\frac{1}{\Gamma(\alpha)}\sum_{k=0}^{m}\Big|\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\frac{f^{(3)}(\theta_k(\tau),u(\theta_k(\tau)))-f^{(3)}(\hat{\theta}_k,u(\hat{\theta}_k))}{3!} \\
&(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\mathrm{d}\tau\Big| \\
&\leqslant \frac{M_2\Delta t}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\left|(\tau-t_{2k})(\tau-t_{2k+1})(\tau-t_{2k+2})\right|\mathrm{d}\tau \\
&\leqslant \frac{M_2\Delta t^4}{\Gamma(\alpha)}\sum_{k=0}^{m}\int_{t_{2k}}^{t_{2k+2}}(t_{2m+2}-\tau)^{\alpha-1}\mathrm{d}\tau \\
&\leqslant \frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}t_{2m+2}^{\alpha}, \qquad (3.17)
\end{aligned}
$$

这里 $M_2=\sup\limits_{t\in[0,T]}\left|f^{(4)}(t,u(t))\right|$.

将（3.16）-（3.17）代入到（3.15），可得：

$$
\left|r_{2m+2}(\Delta t)\right|\leqslant \frac{2^{\alpha+2}\Delta t^{\alpha+3}M_1}{\Gamma(\alpha+2)}\sum_{k=0}^{m}|\beta_k|+\frac{M_2\Delta t^4}{\alpha\Gamma(\alpha)}t_{2m+2}^{\alpha}.
$$

根据引理 3.1，当 $0<\alpha\leqslant 1$ 时，有：

$$
\begin{aligned}
\left|r_{2m+2}(\Delta t)\right| &\leqslant C\frac{2^{\alpha+2}M_1}{\Gamma(\alpha+2)}\Delta t^{\alpha+3}+\frac{M_2}{\alpha\Gamma(\alpha)}T^{\alpha}\Delta t^4 \\
&\leqslant C\Delta t^{\alpha+3},
\end{aligned}
$$

这里 C 仅依赖于 M_1, M_2, α 和 T.

当 $\alpha > 1$ 时，得：

$$
\begin{aligned}
\left|r_{2m+2}(\Delta t)\right| &\leqslant C\frac{2^{\alpha+2}M_1}{\Gamma(\alpha+2)}\Delta t^{\alpha+3}m^{\alpha-1}+\frac{M_2T^{\alpha}}{\alpha\Gamma(\alpha)}\Delta t^4 \\
&= C\frac{2^3M_1}{\Gamma(\alpha+2)}\Delta t^4(2m\Delta t)^{\alpha-1}+\frac{M_2}{\alpha\Gamma(\alpha)}T^{\alpha}\Delta t^4 \\
&\leqslant C\frac{2^3M_1}{\Gamma(\alpha+2)}T^{\alpha-1}\Delta t^4+\frac{M_2}{\alpha\Gamma(\alpha)}T^{\alpha}\Delta t^4 \\
&\leqslant C\Delta t^4.
\end{aligned}
$$

引理 3.3 证明完毕.

由引理 3.2 和引理 3.3 知，我们所构造的格式的截断误差为：当 $0<\alpha\leqslant 1$ 时，是 $3+\alpha$ 阶；当 $\alpha>1$ 时，是 4 阶. 亦即，截断误差 $r_n(\Delta t), n=1,2,\cdots,2N$, 满足：

$$
r_n(\Delta t) \leqslant \begin{cases} C\Delta t^{3+\alpha},\ 0<\alpha\leqslant 1, \\ C\Delta t^4,\ \alpha>1. \end{cases} \tag{3.18}
$$